JOURNAL OF GUIDANCE, CONTROL, AND DYNAMICS
Vol. 32, No. 1, January–February 2009

Engineering Notes

ENGINEERING NOTES are short manuscripts describing new developments or important results of a preliminary nature. These Notes should not exceed 2500 words (where a figure or table counts as 200 words). Following informal review by the Editors, they may be published within a few months of the date of receipt. Style requirements are the same as for regular contributions (see inside back cover).

Time-Optimal Low-Thrust Formation Maneuvering Using a Hybrid Linear/Nonlinear Controller

Balaji Shankar Kumar* and Alfred Ng†
Canadian Space Agency,
St. Hubert, Quebec J3Y 8Y9, Canada

DOI: 10.2514/1.36024

Nomenclature

a = acceleration
k = arbitrary gain constant
k_d = linear derivative gain
k_p = linear proportional gain
t = time
u = applied control
x = radial distance
x_m = average radial distance
y = along-track distance
y_m = average along-track distance
z = cross-track distance
ω = mean motion

I. Introduction

MAINTAINING two spacecraft in a formation with a conventional fuel-based propulsion system could be challenging in its own right. But this problem becomes more difficult when the propulsion system has limited capabilities or control actuation is available in only one or two axes and time-optimal and fuel-optimal control is desired. Fortunately, for the spacecraft relative motion problem, because the relative equations of motion are coupled, motion in more than one direction can be controlled by control actuation in only one direction. In this Note, we formulate a hybrid linear/nonlinear controller that can efficiently maneuver a spacecraft formation by applying control only in the along-track direction. We also assume that the available control is very small in magnitude. This type of problem could be anticipated when dealing with low-thrust systems such as plasma thrusters, control with differential drag or solar radiation pressure, or any other system with very-low-thrust capability.

One of the very early papers on underactuated control of a spacecraft formation with low-thrust capability by Leonard [1] presented an elegant algorithm to control a formation with only differential drag (assumed to be a constant and acting only in the along-track direction). Although the derived control was proven to be time-optimal control, it was not fuel-optimal. This was because differential drag was assumed to be a free resource with no need to economize it. However, for a spacecraft relying on expendables for control (plasmic thrusters), fuel economy would be highly desired.

Some of the recent papers on this topic [2,3] also address the problem of spacecraft formation control with limited resources. The idea is to use linear or nonlinear controllers with a saturation function. The resultant controllers are robust and globally stable but not time-optimal. In the applied control field, the problem to control less frequently and more efficiently is more often an issue and elegant solutions exist for the same problem [4,5]. In this Note, we borrow some ideas discussed from [1,5] and apply them to the spacecraft formation-control problem. The result is a time-optimal controller that is easy to implement for underactuated formation control with limited resources. Although the controller is not proven to be fuel-optimal, it is shown to consume less fuel than a conventional time-optimal or a linear robust controller.

II. Derivation of System Equations

A convenient approach for satellites in formation is to use a Cartesian rotating coordinate system with the origin of the reference frame attached to a leader satellite moving in a reference orbit, as shown in Fig. 1. Here, the x axis is oriented in the radial direction pointing away from the Earth, the z axis is pointed parallel to the angular momentum vector, and the y axis completes the right-handed coordinate system. In the case of a circular reference orbit, the y axis is parallel to the velocity vector of the leader satellite.

Note that the coordinate system will be rotating with a constant angular velocity equal to the mean motion ω for a circular reference orbit. The Hill–Clohessy–Wiltshire (HCW) equations that describe the relative motion are given as [6]

$$\ddot{x} - 2\omega\dot{y} - 3\omega^2 x = a_x \tag{1}$$

$$\ddot{y} + 2\omega\dot{x} = a_y \tag{2}$$

$$\ddot{z} + \omega^2 z = a_z \tag{3}$$

The equations for the unforced positions and accelerations, in terms of initial conditions and accelerations, are given next in state variable form:

$$\begin{bmatrix} x \\ y \\ \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 4-3\cos\omega t & 0 & \frac{1}{\omega}\sin\omega t & -\frac{2}{\omega}\cos\omega t + \frac{2}{\omega} \\ 6\sin\omega t - 6\omega t & 1 & \frac{2}{\omega}\cos\omega t - \frac{2}{\omega} & \frac{4}{\omega}\sin\omega t - 3t \\ 3\omega\sin\omega t & 0 & \cos\omega t & 2\sin\omega t \\ 6\omega\cos\omega t - 6\omega & 0 & -2\sin\omega t & 4\cos\omega t - 3 \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \\ \dot{x}_0 \\ \dot{y}_0 \end{bmatrix} \tag{4}$$

The solution to the HCW equations for a constant applied force can be found by using the method of variation of constants [7]:

Received 4 December 2007; revision received 7 July 2008; accepted for publication 8 July 2008.

*Visiting Scientist, Control and Analysis Group, Spacecraft Engineering.
†Manager, Control and Analysis Group, Spacecraft Engineering.

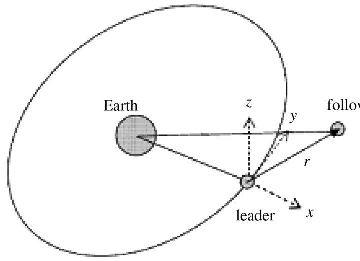

Fig. 1 Hill's frame of reference.

$$\mathbf{x}\,(t) = \phi\mathbf{x}_0 + \int_{t_0}^{t}\left(\begin{bmatrix} \frac{1}{\omega}\sin\omega t \\ \frac{2}{\omega}\cos\omega t - \frac{2}{\omega} \\ \cos\omega t \\ -2\sin\omega t \end{bmatrix} da_x + \begin{bmatrix} -\frac{2}{\omega}\cos\omega t + \frac{2}{\omega} \\ \frac{4}{\omega}\sin\omega t - 3t \\ 2\sin\omega t \\ 4\cos\omega t - 3 \end{bmatrix} da_y\right) \mathrm{d}\tau \tag{5}$$

where

$$\phi = \begin{bmatrix} 4 - 3\cos\omega t & 0 & \frac{1}{\omega}\sin\omega t & -\frac{2}{\omega}\cos\omega t + \frac{2}{\omega} \\ 6\sin\omega t - 6\omega t & 1 & \frac{2}{\omega}\cos\omega t - \frac{2}{\omega} & \frac{4}{\omega}\sin\omega t - 3t \\ 3\omega\sin\omega t & 0 & \cos\omega t & 2\sin\omega t \\ 6\omega\cos\omega t - 6\omega & 0 & -2\sin\omega t & 4\cos\omega t - 3 \end{bmatrix} \tag{6}$$

and da_x and da_y are the differential accelerations in the radial and the along-track directions.

Assuming da_x and $da_z = 0$, and denoting da_y as u (the applied control) in Eq. (5) and evaluating it gives

$$x = 4x_0 - 3\cos\omega t x_0 + \frac{1}{\omega}\sin\omega t\dot{x}_0 - \frac{2}{\omega}\cos\omega t + \frac{2}{\omega}\dot{y}_0 + 2u\sin\omega t + \frac{2}{\omega}ut \tag{7}$$

$$y = 6x_0\sin\omega t - 6x_0\omega t + y_0 + \frac{2}{\omega}\dot{x}_0\cos\omega t - \frac{2}{\omega}\dot{x}_0 + \frac{4}{\omega}\dot{y}_0\sin\omega t - 3\dot{y}_0 t + 4\cos\omega t - \frac{3}{2}ut^2 \tag{8}$$

$$\dot{x} = 3x_0\omega\sin\omega t + \dot{x}_0\cos\omega t + 2\dot{y}_0\sin\omega t + 2u\omega\cos\omega t \tag{9}$$

$$\dot{y} = 6x_0\omega\cos\omega t - 6x_0\omega - 2\dot{x}_0\sin\omega t + 4\dot{y}_0\cos\omega t - 3\dot{y}_0 - 4u\omega\sin\omega t - 3ut \tag{10}$$

From Eqs. (7) and (8) it can be seen that the average value of x over time is

$$x_m = 4x_0 + \frac{2\dot{y}_0}{\omega} + \frac{2ut}{\omega} \tag{11}$$

and that the average value of y over time is

$$y_m = y_0 - \frac{2\dot{x}_0}{\omega} - 3\left(4x_0 + \frac{2\dot{y}_0}{\omega}\right)\omega t - \frac{3}{2}ut^2 \tag{12}$$

It can be noted from Eq. (7) that

$$4x = -\left(12x_0 + 8\frac{\dot{y}_0}{\omega}\right)\cos\omega t + \left(4\frac{\dot{x}_0}{\omega} - 8\frac{a}{\omega^2}\right)\sin\omega t + \left(16x_0 + 8\frac{\dot{y}_0}{\omega}\right) + \frac{8ut}{\omega} \tag{13}$$

and from Eq. (10) that

$$\frac{2\dot{y}}{\omega} = \frac{4}{\omega}\left(\frac{2u}{\omega} - \dot{x}_0\right)\sin\omega t + \frac{4}{\omega}(3x_0\omega + 2\dot{y}_0)\cos\omega t - \frac{6}{\omega}(2x_0\omega + \dot{y}_0) - \frac{6ut}{\omega} \tag{14}$$

Summing and combining the terms of Eqs. (13) and (14) gives

$$4x + \frac{2\dot{y}}{\omega} = 4x_0 + \frac{2\dot{y}_0}{\omega} + \frac{2ut}{\omega} \tag{15}$$

Similarly, it can be noted from Eq. (8) that

$$y = \left(\frac{2\dot{x}_0}{\omega} - \frac{4u}{\omega^2}\right)\cos\omega t + \left(6x_0 + \frac{4\dot{y}_0}{\omega}\right)\sin\omega t + y_0 - \left(\frac{2\dot{x}_0}{\omega} - \frac{4u}{\omega^2}\right) - \left(6x_0 + \frac{3\dot{y}_0}{\omega}\right)\omega t - \frac{3}{2}ut^2 \tag{16}$$

and from Eq. (9) that

$$-\frac{2\dot{x}}{\omega} = \left(-6x_0 - \frac{4\dot{y}_0}{\omega}\right)\sin\omega t + \left(-\frac{2\dot{x}_0}{\omega} + \frac{4u}{\omega^2}\right)\cos\omega t - \frac{4u}{\omega^2} \tag{17}$$

Adding Eqs. (16) and (17) gives

$$y - \frac{2\dot{x}}{\omega} = y_0 - \frac{2\dot{x}_0}{\omega} - 3\left(4x_0 + \frac{2\dot{y}_0}{\omega}\right)\omega t - \frac{3}{2}ut^2 \tag{18}$$

Combining Eqs. (11) and (15) gives the average value for x:

$$x_m = 4x + \frac{2\dot{y}}{\omega} \tag{19}$$

Similarly combining Eqs. (12) and (18) gives the average value of y:

$$y_m = y - \frac{2\dot{x}}{\omega} \tag{20}$$

It can be seen from Eqs. (19) and (20) that the average values of x and y can be calculated from the instantaneous relative positions and velocities. With the average values for x and y known, α is defined as the difference between the actual position x and the average position x_m for that point of time,

$$\alpha = x - x_m \tag{21}$$

and β is defined as the difference between the actual position y and the average position y_m for that point of time:

$$\beta = y - y_m \tag{22}$$

This definition of actual, average, and the difference is better illustrated in Fig. 2.

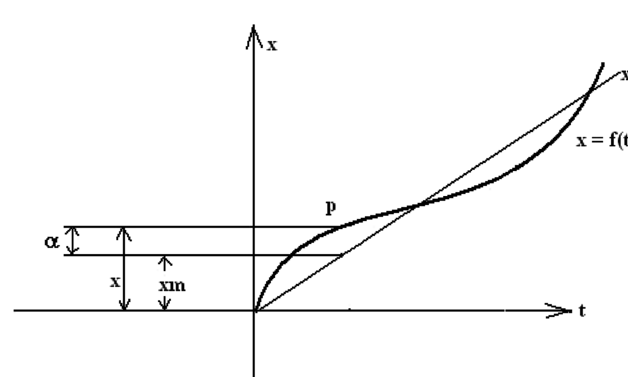

Fig. 2 Illustration of actual and average positions.

Rearranging Eq. (21) and substituting it in Eq. (19) gives

$$\alpha = -3x - 2\frac{\dot{y}}{\omega} \tag{23}$$

Similarly rearranging Eq. (22) and substituting it in Eq. (20) gives

$$\beta = \frac{2\dot{x}}{\omega} \tag{24}$$

Rearranging and differentiating Eq. (22) in terms of y_m twice gives

$$\ddot{y}_m = \ddot{y} - \ddot{\beta} \tag{25}$$

Now substituting Eq. (2) in Eq. (25) and denoting u as the applied control gives

$$\ddot{y}_m + \ddot{\beta} = u - \omega^2\beta \tag{26}$$

Differentiating Eq. (24) twice,

$$\ddot{\beta} = \frac{2\ddot{x}}{\omega} \tag{27}$$

Differentiating Eq. (1) and substituting in for the value of $\ddot{x}$ gives

$$\ddot{\beta} = 4\ddot{y} + 6\omega\dot{x} \tag{28}$$

Now differentiating Eq. (24) and substituting the value of $\dot{x}$ in Eq. (28) yields

$$\ddot{\beta} = \ddot{y} + 3u \tag{29}$$

Equation (29) can be rewritten in terms of y_m as

$$\ddot{y}_m = -3u \tag{30}$$

Noting that $\dot{y}_m$ is coupled with x as

$$\dot{y}_m = -\frac{3x_m\omega}{2} \tag{31}$$

will drive states y_m and x_m to the origin by driving y_m and $\dot{y}_m$ to the origin using system equation (30).

The use of average values of the relative states for control is a significant step in reducing the complexity of the time- and fuel-optimal control problems. This is due to the fact that only the secular part of the relative motion is controlled and any cyclic part caused by perturbations such as J2 is ignored, thereby reducing unwanted control actuation and fuel expenditure. The practical realization of the average values is not difficult with an implementation of a low-pass filter.

III. Hybrid Linear/Nonlinear Time-Optimal Control Law

The double-integrator plant represented as Eq. (30) can be written in the state space form as

$$[\mathbf{y}] = \begin{bmatrix}\dot{y}_1\\ \dot{y}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\begin{bmatrix}y_m\\ \dot{y}_m\end{bmatrix} + \begin{bmatrix}0\\ -3\end{bmatrix}u \tag{32}$$

where $y_1 = y_m$ and $y_2 = \dot{y}_1$.

The preceding system is inherently unstable because a bounded control input could produce an unbounded output. The open-loop transfer function has a pair of poles at the origin, $s = 0$, of the complex plane. The time-optimal discontinuous bang–bang controller for the preceding system can be derived using Pontryagin's minimum principle and is given as [8]

$$U_1(\mathbf{y}) = \begin{cases} u_{\max}[y_m - S_{1,2}] & \text{if } y_m - S_{1,2} \neq 0\\ u_{\max}\mathrm{sgn}[\dot{y}_m] & \text{if } y_\mathrm{m} - S_{1,2} = 0\end{cases} \tag{33}$$

where

$$S_{1,2} = \frac{|\dot{y}_m|\dot{y}_m}{6u_{\max}} \tag{34}$$

We make the preceding control law linear near the origin and nonlinear away from the origin by using a method as discussed in [5]

The control law given in Eq. (33) can be made continuous by modifying it as

$$U_2(\mathbf{y}) = \mathrm{sat}\{k_p[y_m - S_{1,2}]\} \tag{35}$$

We will name k_p as the proportional gain constant. The saturation function is defined as

$$\mathrm{sat}\,(\alpha) = \left\{\begin{array}{c} +u_{\max} \quad \text{if } \alpha \geq +1\\ \alpha \quad \text{if } |\alpha| < 1\\ -u_{\max} \quad \text{if } \alpha \leq -1\end{array}\right\} \tag{36}$$

When $\dot{y}_m$ is close to zero and $|k_p y_m| < 1$, the control becomes $U_2(\mathbf{x}) = k_p y_m$. Substituting this in Eq. (30), we see that

$$\ddot{y}_m + 3k_p y_m = 0 \tag{37}$$

Adding a rate term to the control equation helps to add some damping to the system, especially when $\mathbf{y} \approx 0$:

$$U_3(\mathbf{x}) = k_p y_m + k_d\dot{y}_m \tag{38}$$

Ideally, we want to have a time-optimal control law that is a combination of nonlinear switching far from the origin and a linear controller near the origin. The two conditions could be satisfied if we modify the control law equation (35) to

$$U_4(\mathbf{y}) = \mathrm{sat}\left\{k_p[y_m - S_{1,2}] + k\mathrm{sat}\left(\frac{k_d\dot{y}_m}{k}\right)\right\} \tag{39}$$

where k is an arbitrary constant.

Ideally, k should be small enough to make $U_4(y) \approx U_2(y)$, when $k_d\dot{y}_m/k > 1$ and y_m and $\dot{y}_m$ are not close to the origin. For $k_d\dot{y}_m/k < 1$, then $U_4(y) \approx U_3(y)$. The linear gains k_p and k_d can be chosen by using one of the standard gain selection techniques such as the pole-placement method or linear robust design methodology. For positive linear gains k_p and k_d, the controller is shown to be globally stable in [5].

IV. Controller Performance and Validation

In this section, the control performance of the three different controllers [namely, the time-optimal bang–bang controller, the linear saturated controller, and the hybrid controller (Eqs. (33), (38), and (39), respectively)] are compared and validated. The linear gains for the linear and the hybrid controller were selected based on the pole-placement method, as shown here.

Substitution of Eq. (38) in Eq. (32) yields

$$\begin{bmatrix}\dot{y}_1\\ \dot{y}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\begin{bmatrix}y_m\\ \dot{y}_m\end{bmatrix} + \begin{bmatrix}0\\ -3\end{bmatrix}\begin{bmatrix}k_p & k_d\end{bmatrix}\begin{bmatrix}y_m\\ \dot{y}_m\end{bmatrix} \tag{40}$$

Simplification of Eq. (40) yields

$$\begin{bmatrix}\dot{y}_1\\ \dot{y}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ -3k_p & -3k_d\end{bmatrix}\begin{bmatrix}y_m\\ \dot{y}_m\end{bmatrix} \tag{41}$$

The characteristic equation is given by

$$D(s) = \det(sI - A) \tag{42}$$

where

$$A = \begin{bmatrix}0 & 1\\ -3k_p & -3k_d\end{bmatrix} \tag{43}$$

$$D(s) = s^2 + 3k_d s + 3k_p \tag{44}$$

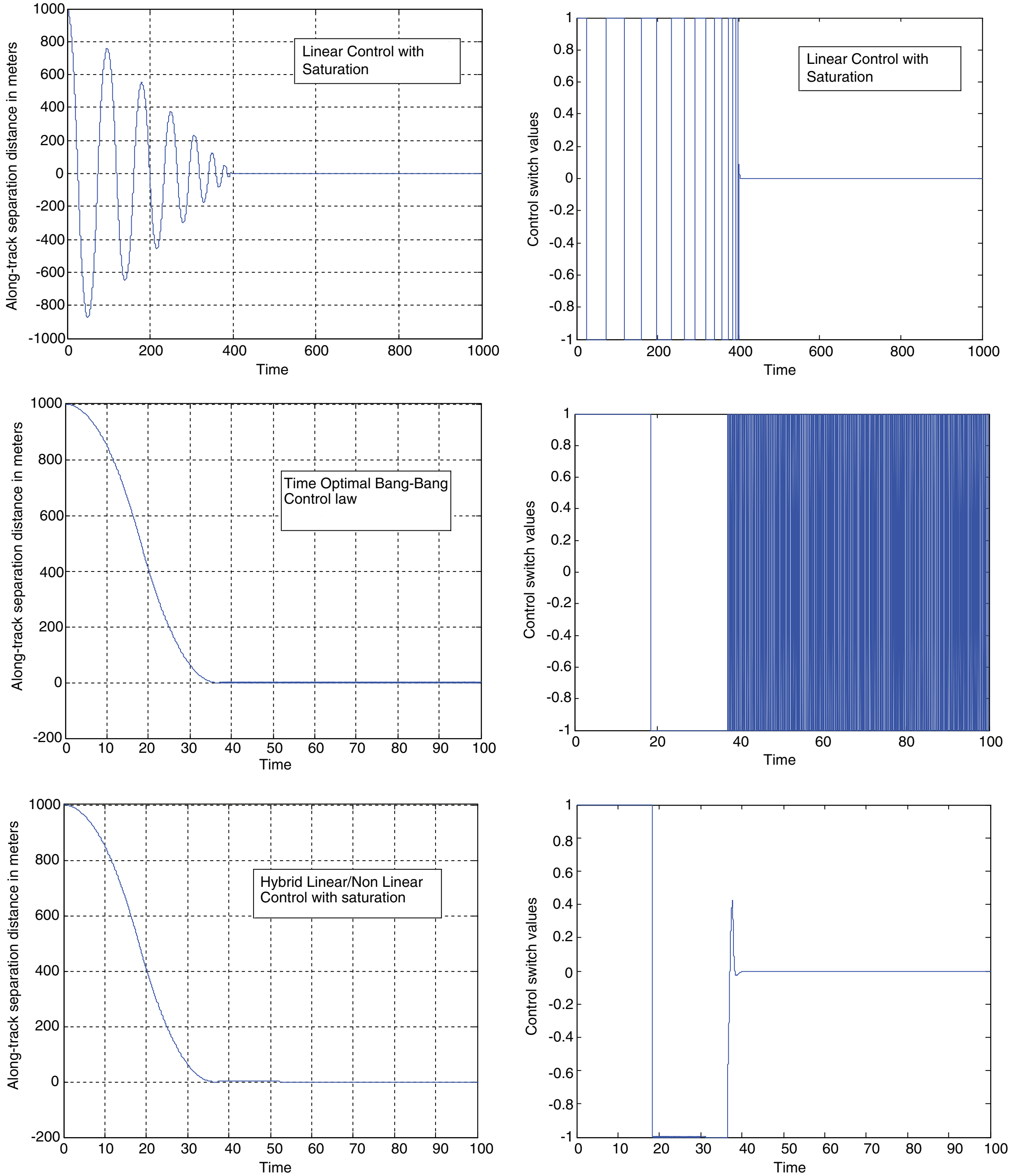

Fig. 3 Error response and control history of the following controllers in order: saturated linear, time-optimal bang–bang, and time-optimal hybrid linear/nonlinear.

From Eq. (44), it can be seen that it is possible to choose the linear gains k_p and $k_d > 0$ such that the roots are negative. As an example, if the roots were desired to be $s = -2$ and -3, then $k_p = 2$ and $k_d = 5/3$. The error responses of the three different control laws (33), (38), and (39) and their corresponding control usage are plotted and shown in Fig. 3 for the following initial conditions: $y_m = 1000$, $\dot{y}_m = 0$, $k_p = 2$, $k_d = 5/3$, $k = 0.2$, and $-1 \leq u \leq 1$.

The control usage of the hybrid controller can be seen to be the least of the three controllers. Similarly, the transient time is almost similar to that of the time-optimal bang–bang controller with only little difference near the origin due to the operation of the controller as a linear system with the rapidly diminishing effect of the $\dot{y}_m|\dot{y}_m|$ term. Clearly, the hybrid controller excels over the time-optimal bang–bang in terms of control usage and the linear controller saturated in terms of response time.

V. Conclusions

A hybrid linear/nonlinear controller was formulated that combines the efficacies of linear control and time-optimal bang–bang control methodologies to efficiently maneuver spacecraft in formation. The system equations are formulated based on the time-averaged values of the relative coordinates rather than their instantaneous values. This formulation is very appropriate for low-thrust fuel-optimal problems in which unwanted fuel actuation on cyclic disturbances is to be minimized. The resultant system equation is a double-integrator system with the control variable only in the along-track direction of motion. The control equation requires the feedback information of only two parameters: namely, the average along-track position and the rate to drive the along-track and radial positions and rates to the origin. The error responses and the time histories of the hybrid linear/nonlinear control law are compared with those of the linear saturated control law and time-optimal bang–bang control law. The overall performance of the formulated hybrid linear/nonlinear control law is shown to be better in terms of control usage and transient time. The hybrid controller presented in this Note will be used as one of the formation maneuvering controllers in the upcoming JC2SAT mission [9].

References

[1] Leonard, C. L., "Orbital Formation-Keeping with Differential Drag," *Journal of Guidance, Control, and Dynamics*, Vol. 12, No. 1, 1989, pp. 108–113.
doi:10.2514/3.20374

[2] Gurfil, P., Kasdin, N. J., "Nonlinear Low-Thrust Lyapunov-Based Control of Spacecraft Formations," *Proceedings of the American Control Conference*, American Automatic Control Council, Evanston, IL, June 2003, pp. 1758–1763.

[3] Starin, S. R., Yedavalli, R. K., Sparks, A. G., "Design of a LQR Controller of Reduced Inputs for Multiple Spacecraft Formation Flying," *Proceedings of the American Control Conference*, American Automatic Control Council, Evanston, IL, June 2001, pp. 118–126.

[4] Rauch, L. L., and Howe, R. M., "A Servo with Linear Operation in a Region About the Optimum Discontinuous Switching Curve," *Symposium on Nonlinear Circuit Analysis, Proceedings of the IRE*, Vol. 44, No. 3, Mar. 1956, pp. 429–430.
doi:10.1109/JRPROC.1956.275113

[5] Kalyon, M., "Design of Continuous Time Controllers Having Almost Minimum Time Response," *Journal of Dynamic Systems, Measurement, and Control*, Vol. 124, June 2002, pp. 252–260.
doi:10.1115/1.1468862

[6] Chobotov, V., *Orbital Mechanics*, 3rd edition, AIAA Education Series, AIAA, Reston, VA, 2002, pp. 158–159.

[7] Fehse, W., *Automated Rendezvous and Docking of Spacecraft*, Cambridge Aerospace Series, Cambridge Univ. Press, New York, 2003, pp. 40–41.

[8] Desineni, S. Naidu, *Optimal Control Systems*, CRC Press, Boca Raton, FL, 2002, pp. 315–328.

[9] Yoshihara, K., Mierlo, M., Ng, A., Shankar Kumar, B., De Ruiter, A., and Komatsu, Y., "JC2SAT-FF: An International Collaboration Nano-SAT Project Overview of the System Analyses and Design," *3rd International Symposium on Formation Flying, Missions and Technologies*, ESA, European Space Research and Technology Centre, Noordwijk, The Netherlands, May 2008.